Topology in Analysis

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Quick Review of Sobolev Maps

The **weak derivative** of a locally intgegrable function $u : B \to \mathbb{R}^2$ is $\nabla u : B \to \mathbb{R}^{2\times 2}$ satisfying

$$\int_B u \cdot \nabla\varphi \, \mathrm{d}x = -\int_B \varphi \cdot \nabla u \, \mathrm{d}x \qquad \forall \varphi \in C_c^\infty(B; \mathbb{R}^2)$$

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A function $u : B \to \mathbb{R}^2$ is in $W^{1,2}(B;\mathbb{R}^2)$ if it is weakly differntiable and $\|u\|_2$, $\|\nabla u\|_2$ are both finite.

Energy Functional & Admissibles

The Dirichlet energy functional $\mathbb{D}(u)$ can be used to describe the strain energy density under a deformation u.

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We shall consider the following space of admissible maps

$$\mathcal{A} = \{ u \in u_2 + W_0^{1,2}(B; \mathbb{R}^2) : \det \nabla u = 1 \text{ a.e. } \}$$

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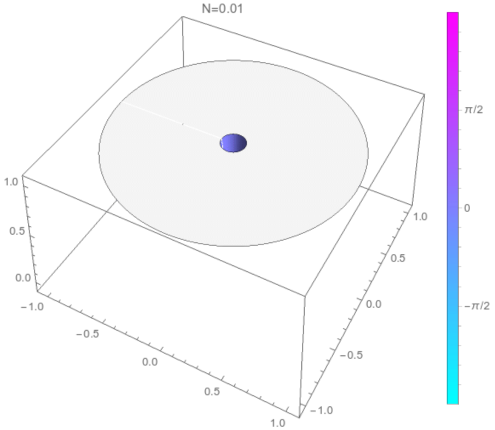
The chosen boundary condition is the **double covering map** $u_2 : B \to B' := \frac{1}{\sqrt{2}} B$ which is given (in complex coordinates) by

$$r \mapsto \frac{1}{\sqrt{2}} r \qquad \theta \mapsto 2\theta$$

Plot of u_2



Variations & Optimisation

For a domain $\Omega \subset \mathbb{R}^2$, we define

$$\mathcal{V}(\Omega) := \{\varphi \in \mathrm{id} + W_0^{1,2}(\Omega;\mathbb{R}^2) : \det \nabla\varphi = 1 \text{ a.e. } \}$$

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Then a **mixed variation** of u_2 is a map of the form

$$\psi \circ u_2 \circ \varphi \in \mathcal{A} \qquad \varphi \in \mathcal{V}(B) \quad \psi \in \mathcal{V}(B')$$

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But just how big/small is this class?

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The Topological Degree

Consider a map $u \in C^1(B;\mathbb{R}^2)$ and a regular point $y \notin u(\partial B)$. The **topological degree** is given by

$$\deg(u, B, y) = \sum_{x \in u^{-1}\{y\}} \operatorname{sgn} \det \nabla u(x)$$

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We also have that the degree is constant on connected components of $u(B)$.

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What about a general admissible map?

Counting Invertible Points

Consider $u \in \mathcal{A}$ such that $\left|u^{-1}\{y\}\right| < \infty$. It turns out that

$$\left|u^{-1}\{y\}\right| \leq \deg(u, B, y) \quad \forall y$$

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We then define the **associated self-map** $\mathrm{id}_u : B \to B$ by

$$\mathrm{id}_u(x) = \begin{cases} x' & u^{-1}(u(x)) = \{x, x'\} \\ x & u^{-1}(u(x)) = \{x\} \end{cases}$$

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How many are there?

Lefschetz-Hopf Theorem

The **Lefschetz number** is

$$
\begin{aligned}
\Lambda_{\mathrm{id}_u} &= \sum_{k \geq 0} (-1)^k \operatorname{tr}\left((\mathrm{id}_u)_* | H_k(B; \mathbb{Q})\right) \\
&= \sum_{k=0} (-1)^k \operatorname{tr}\left((\mathrm{id}_u)_* | H_k(B; \mathbb{Q})\right) && (B \text{ contractible}) \\
&= \operatorname{tr} \mathrm{id} = 1 && (B \text{ connected})
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The **Lefschetz-Hopf** theorem for ENR's states that

$$\Lambda_{\mathrm{id}_u} = \sum_{x\in \mathrm{Fix}(\mathrm{id}_u)} i(\mathrm{id}_u, x) = \sum_{x\in \mathrm{Fix}(\mathrm{id}_u)} 1 = |\mathrm{Fix}(\mathrm{id}_u)|$$

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Result: Any orientation preserving injective map self-map of the ball has either 1 fixed point or infinite fixed points.

Hadamard-Cacciopoli Theorem

We now know that any $u \in \mathcal{A}$ such that $\left|u^{-1}\{y\}\right| < \infty$ satisfies

- It is 1:1 somewhere in its domain.
- It is 2:1 everywhere else.

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This is interesting when compared to the C^1 case.

If $u \in C^1(X;Y)$ is proper for X connected and Y simply connected, then

$$\det \nabla u \neq 0 \Rightarrow u \text{ is a diffeomorphism}$$

so a discontinuity in the derivative is essential for 2:1 behaviour.

Elephant in the Room

What if there exists y such that $\left|u^{-1}\{y\}\right| = \infty$? Either countable or uncountable?

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In the uncountable case, we could have a line in B that is compressed to a point in the image.

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Proposed Construction

We want to construct a map that has gradients in $SL(2)$ that can map a curve onto a point.

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We consider maps on an annulus $B_{r_n} - B_{r_{n+1}}$ of the form

$$u_n(x) = \ell_n(x) R(\omega_n(x)) x$$

Here ℓ_n are scalar functions governing the contracting factor and ω_n are scalar functions governing rotation.

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The r_n must be a decreasing sequence in $[0, 1]$ that is bounded away from zero and has initial condition $r_0 = 1$.

Derived PDE

We calculate the Jacobian using polar coordinates

$$\det \nabla u_n = (r\ell_{n,r} + \ell_n)\ell_n(\omega_{n,\theta} + 1) - r\ell_{n,\theta}\ell_n^2\omega_{n,r}$$

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If we assume $\ell_n = \ell_n(r)$ then the Jacobian constraint is just

$$\omega_{n,\theta} = \left(\frac{1}{(r\ell_{n,r} + \ell_n)\ell_n} - 1\right)$$

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$$\begin{aligned} \omega_{n,\theta} &= \left(\frac{1}{(r\ell_{n,r} + \ell_n)\ell_n} - 1\right) \\ \Rightarrow \omega &= \left(\frac{1}{(r\ell_{n,r} + \ell_n)\ell_n} - 1\right)\theta + C_n(r) \end{aligned}$$

for arbitrary functions C_n, which we shall set to zero.

Boundary Conditions

For u to be continuous, we must have that a change in a multiple of 2π in θ results in a change of a multiple of 2π in ω.

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This leads us to the following ODE for ℓ_n:

$$(r\ell_{n,r} + \ell_n)\ell_n = 1$$

which has solution

$$\ell_n(r) = \frac{\sqrt{c_n + r^2}}{r}$$

for some constants c_n.

Continuity & Contracting Factors

To ensure continuity we must have

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Combining these with the equation for ℓ_n in the previous slide, we obtain

$$\ell_n(r) = \frac{1}{r}\sqrt{r_n^2\left(\left(\frac{r_n}{\alpha_n}\right)^2 - 1\right) + r^2}$$

$$r_n = \frac{\alpha_n}{\sqrt{2}}\sqrt{1 \pm \sqrt{1 + \left(\frac{2}{\alpha_n \alpha_{n-1}}\right)^2 (\alpha_{n-1}^2 - r_{n-1}^2) r_{n-1}^2}}, \quad r_0 = 1$$

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Unfortunately, this sequence is increasing for α_n increasing (proof left as exercise).

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Make the ℓ_n depend on θ?

Try to contract a ray rather than a circle?