Mean Hadamard Inequalities & Elasticity

Elliott Sullinge-Farrall

University of Surrey

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Hadamard's Inequality

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From now on, we will consider only the case of $n = 2$ dimensions.

Basic Properties

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This can be used to obtain our first result:

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We finally note that, since $\mathbb{E}_p$ is degree 2 homogeneous, non-negativity is equvalent to existence of a global minimiser.

Elasticity

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However, in general, these minimisers will not be mass conserving:

$$\det \nabla u \neq 1$$

Mass Conservation

We now introduce the space of mass conserving admissibles:

$$\mathcal{A} = \{u \in H^1_{u_0}(\Omega; \mathbb{R}^2) : \det \nabla u = \det \nabla u_0 = 1\}$$

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We find that

$$\mathcal{V} = \{\varphi \in H^1_0(\Omega; \mathbb{R}^2) : \det \nabla \varphi = -\operatorname{cof} \nabla u_0 \cdot \nabla \varphi\}$$

Derivation of the Excess

We then find that

$$\mathbb{D}(u_0 + \varphi) = \mathbb{D}(u_0) + \mathbb{E}_{\mathsf{p}}(\varphi) \qquad \forall \varphi \in \mathcal{V}$$

where p solves $\Delta u_0 + \frac{1}{2}\,\mathrm{cof}(\nabla u_0)\nabla \mathsf{p} = 0$.

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where p solves $\Delta u_0 + \frac{1}{2}\,\mathrm{cof}(\nabla u_0)\nabla \mathsf{p} = 0$.

Hence $\mathbb{E}_{\mathsf{p}} \geq 0$ is equivalent to the minimisation of the elastic energy (w.r.t to these variations).

Convexity of the Excess

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The excess, however, is only **polyconvex**.

Polyconvexity (2D)

A function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is polyconvex if there exists convex $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that

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This motivates the search for novel techniques to analyse these functionals.

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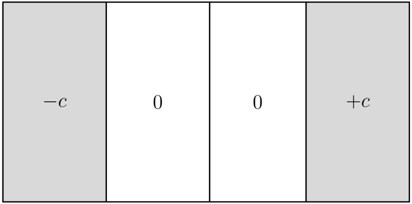
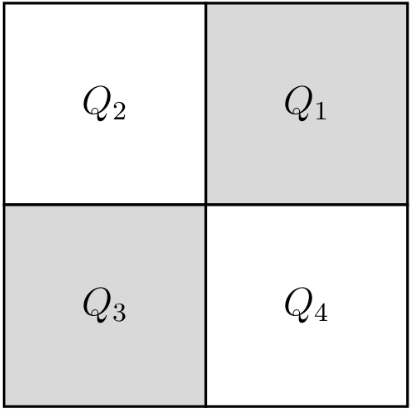
They also considered three state pressure with either 'insulation' or 'point-contact'.

'Window' vs. 'Grid'





Above we have an example of a 'window' layout pressure function.

On the right is an example of a 'grid' layout pressure function.

'Window' vs. 'Grid'

For the 'window' layout, there exists a $\gamma_0 > 0$ (depending only on the domain) such that

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There are also partial results for finer grids.

Radially Linear Pressure

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For reference purposes, we calculate

$$\mathsf{p}_B = \frac{2}{3}c \quad \Rightarrow \quad |\mathsf{p} - \mathsf{p}_B|_\infty = \frac{2}{3}c$$

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Can we obtain a mean Hadamard inequality with $|c| > 3$?

Fourier Splitting

We shall start by writing φ as a Fourtier series:

$$\varphi = \sum_{j \geq 0} \varphi^{(j)} = \frac{1}{2} A_0(r) + \sum_{j > 0} A_j(r) \cos{(j\theta)} + B_j(r) \sin{(j\theta)}$$

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We then observe that the excess splits over the modes in the following way:

$$\mathbb{E}_{\mathsf{p}}(\varphi) = \sum_{j \geq 0} \int_B \left|\varphi^{(j)}_{,r}\right|^2 + \left|\varphi^{(j)}_{,\tau}\right|^2 + \frac{c}{2} \varphi^{(j)}_{,\tau} \times \varphi^{(j)} \, \mathrm{d}x$$

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We note that, if we replaced $\varphi_{,r}^{(j)}$ with $\varphi^{(j)}$, we would have something that resembles a quadratic form in the integrand.

Weighted Poincaré Inequality

We now make use of the following result, a corollary of a weighted Poincaré inequality:

Poincaré Inequality for Modes

Denote by j_0 the first zero of the Bessel J function. Then

$$\int_B \left|\varphi^{(j)}\right|^2 \,\mathrm{d}x \le \frac{1}{j_0^2} \int_B \left|\varphi^{(j)}_{,r}\right|^2 \,\mathrm{d}x \qquad \forall j \ge 1$$

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where the inequality is sharp.

This allows us to write

$$\mathbb{E}_{\mathsf{p}}(\varphi) \geq \sum_{j \geq 0} \int_B v^{(j)} \cdot M(c) v^{(j)} \,\mathrm{d}x \qquad v^{(j)} = \begin{pmatrix} \left|\varphi^{(j)}\right| \\ \left|\varphi^{(j)}_{,\tau}\right| \end{pmatrix}$$

Sufficient Condition for Non-Negativity

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Hence, we have

$$c \leq 4j_0 \quad \Rightarrow \quad \mathbb{E}_{\mathsf{p}}(\varphi) \geq 0$$

For reference, $\frac{2}{3} \times 4j_0 \approx 6.41$

Necessary Conditions

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The 'better' our choice of φ, the tighter our bounds will be.

Inspired by the sharpness of the pointwise Hadamard inequality, we consider φ satisfying the PDI:

$$\begin{aligned} \nabla\varphi &\in \mathrm{O}(2) \\ \varphi|_{\partial B} &= 0 \end{aligned}$$

It is known that there is a dense solution space in $W^{1,\infty}$ and that every solution is piecewise affine.

Constructing Solutions

For convenience, we construct a solution on the domain $[-2, +2]^2$ and then scale several copies of the solution to make them fit in the unit ball without overlaps.

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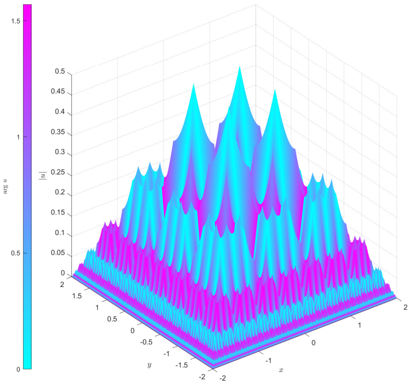
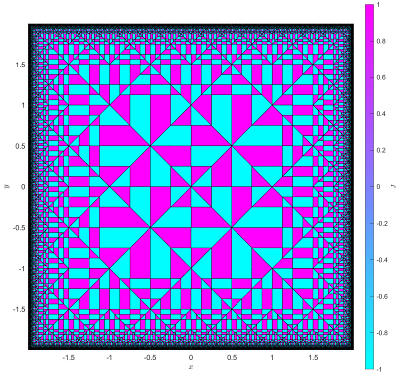
We will obtain different thresholds for different arrangements of the squares.

Construction on a Square

The solution on the square is plotted below:





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Note that any symmetric arrangement will not yield a finite upper bound.

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How does u_0 cause radial symmetry in p? What about regularity?